# Engineering Notes

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## Generalized One-Dimensional Compressible Flow Matrix Inverse

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### Nomenclature

$A$ = cross-sectional area available for flow
$c_p$ = constant pressure specific heat
$D_h$ = hydraulic diameter
$E$ = algebraic term defined by Eq. (2a)
$F$ = algebraic term defined by Eq. (2b)
$F_D$ = drag force other than wall shear
$f$ = Fanning friction factor
$I$ = impulse function
$M$ = Mach number
$\dot{m}$ = mass flow rate
$P$ = static pressure
$P_0$ = stagnation pressure
$r_i$ = cosine of dimensionless injection velocity
$s$ = entropy
$T$ = static temperature
$T_0$ = stagnation temperature
$V$ = velocity
$x$ = spatial coordinate
$\gamma$ = ratio of specific heats
$\rho$ = density
$\psi$ = $1 + [(\gamma-1)/2]M^2$

### Introduction

ONE-DIMENSIONAL compressible flows are usually classified as simple or generalized. A simple compressible flow is one in which only a single effect, area change for example, is considered. The usual simple flows are area change (isentropic), friction (Fanno), heat addition or rejection (Rayleigh), and mass addition. The solutions to simple compressible flows can be expressed as closed-form expressions and are usually tabulated in tables as functions of Mach number and various property ratios. Reference 1 and virtually all compressible flow textbooks provide tables with simple flow tabulations.

If more than a single effect is considered, then a generalized treatment is usually necessary. Unlike simple flows, most generalized flows are not amenable to closed-form analysis. Generalized flows are routinely solved by numerical integration of one or more differential equations involving Mach number and various property ratios. The one-dimensional compressible flow differential equations are generally obtained from tables of influence coefficients. This Note is concerned with a matrix inverse that is useful in devising either the table of influence coefficients or the differential equations for generalized one-dimensional compressible flows.

### Analysis

The formulation of generalized one-dimensional compressible flow in terms of influence coefficients is not new. Shapiro[2] pioneered this type of analysis, and in recent years Zucrow and Hoffman,[3] Beans,[4] and Saad[5] have extended and discussed it. The technique is to write differential equations for a generalized elemental control volume, with area change, heat addition or rejection, friction, and mass addition considered. By applying the conservation equations (mass, linear momentum, and energy), an equation of state (usually the perfect gas law), and definitions of Mach number, the impulse function, stagnation pressure, and entropy, a system of eight differential expressions cast in terms of 12 variables is obtained. Selecting four of the variables, usually called the driving potentials, as known, the system can be expressed in matrix form as

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \gamma M^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dfrac{\frac{\gamma-1}{2}M^2}{\psi} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -\dfrac{\gamma M^2}{1+\gamma M^2} & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -\dfrac{\gamma M^2/2}{\psi} & 1 & 0 & 0 \\ \dfrac{\gamma-1}{\gamma} & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{dP}{P} \\ \dfrac{d\rho}{\rho} \\ \dfrac{dT}{T} \\ \dfrac{dV}{V} \\ \dfrac{dM^2}{M^2} \\ \dfrac{dP_0}{P_0} \\ \dfrac{dI}{I} \\ \dfrac{ds}{c_p} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{d\dot{m}}{\dot{m}} - \dfrac{dA}{A} \\ E + F \\ \dfrac{dT_0}{T_0} \\ \dfrac{dA}{A} \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

where

$$E = -\frac{\gamma M^2}{2}\left(4f\frac{dx}{D_h} + \frac{2dF_D}{\gamma M^2 PA}\right) \tag{2a}$$

$$F = -\gamma M^2 \frac{d\dot{m}}{\dot{m}}(1-r_i) \tag{2b}$$

$$\psi = 1 + \frac{\gamma-1}{2}M^2 \tag{2c}$$

Details of the formal derivation of Eq. (1) can be found in Refs. 2, 3, and 5. The variables are the usual ones associated with compressible flow and are defined in the Nomenclature. The driving potentials are the independent variables and are area change, stagnation temperature change, friction, and mass flow-rate change. The dependent variables are Mach

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number, velocity, density, static pressure, static temperature, impulse, stagnation pressure, and entropy. The dependent variables can be expressed in terms of the driving potentials in differential equation form by solving the system given in Eq. (1). The usual practice is to solve the algebraic system by elimination and to express the results in an influence coefficient table.[2-5]

An alternative to the algebraic solution of the system of differential equations is to form the inverse of the coefficient matrix. Before the advent of symbolic manipulation languages such as MACSYMA, the generation of a symbolic inverse for a $8 \times 8$ matrix was a difficult, tedious undertaking. However, using MACSYMA the task becomes tractable, and, moreover, error free. The MACSYMA-derived inverse for the coefficient matrix in Eq. (1) is given in Eq. (3).

$$\begin{bmatrix} \frac{dP}{P} \\ \frac{d\rho}{\rho} \\ \frac{dT}{T} \\ \frac{dV}{V} \\ \frac{dM^2}{M^2} \\ \frac{dP_0}{P_0} \\ \frac{dI}{I} \\ \frac{ds}{c_p} \end{bmatrix} = \begin{bmatrix} \frac{\gamma M^2}{M^2-1} & -\frac{\gamma(\gamma-1)M^4}{2(M^2-1)} & \frac{\gamma M^2}{M^2-1} & -\frac{1+M^2(\gamma-1)}{M^2-1} & \frac{\gamma M^2\psi}{M^2-1} & 0 & 0 & 0 \\ \frac{1}{M^2-1} & -\frac{(\gamma-1)M^2}{2(M^2-1)} & \frac{M^2}{M^2-1} & -\frac{1}{M^2-1} & \frac{\psi}{M^2-1} & 0 & 0 & 0 \\ \frac{(\gamma-1)M^2}{M^2-1} & -\frac{(\gamma-1)M^2(\gamma M^2-1)}{2(M^2-1)} & \frac{(\gamma-1)M^2}{M^2-1} & -\frac{(\gamma-1)M^2}{M^2-1} & \frac{(\gamma M^2-1)\psi}{M^2-1} & 0 & 0 & 0 \\ -\frac{1}{M^2-1} & \frac{(\gamma-1)M^2}{2(M^2-1)} & -\frac{1}{M^2-1} & \frac{1}{M^2-1} & -\frac{\psi}{M^2-1} & 0 & 0 & 0 \\ -\frac{2\psi}{M^2-1} & \frac{(\gamma M^2-1)\psi}{M^2-1} & -\frac{2\psi}{M^2-1} & \frac{2\psi}{M^2-1} & -\frac{(\gamma M^2+1)\psi}{M^2-1} & 0 & 0 & 0 \\ 0 & \frac{\gamma M^2}{2} & 0 & 1 & -\frac{\gamma M^2}{2} & 0 & 1 & 0 \\ \frac{\gamma M^2}{\gamma M^2+1} & \frac{\gamma M^2}{\gamma M^2+1} & \frac{\gamma M^2}{\gamma M^2+1} & \frac{1}{\gamma M^2+1} & 0 & 1 & 0 & 0 \\ 0 & -\frac{(\gamma-1)M^2}{2} & 0 & -\frac{\gamma-1}{\gamma} & \psi & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{d\dot{m}}{\dot{m}} - \frac{dA}{A} \\ E+F \\ \frac{dT_0}{T_0} \\ \frac{dA}{A} \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

Equation (3) can be used to generate the table of influence coefficients, similar to the ones in Refs. 2-5, or to directly obtain the differential equations for the Mach number and various property ratios. The availability of the inverse allows a straightforward approach to obtaining compressible flow formulations, is useful for pedagogical purposes, and should be available in the compressible flow literature. The generalized one-dimensional compressible flow technique is well known; the purpose of this Note is to make available the inverse in the literature.

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## References

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